THE DERIVATIVE

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First lesson

THE DEFINITION OF THE DERIVATIVE

DEFINITION OF DERIVATIVE

Let $f: [a, b] \to \mathbb{R}$ be a function, $x_0 \in (a, b)$.

f is said to be <u>differentiable</u> at x_0 if the limit:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists and is finite.

Such limit, provided it exists and is finite, is called the <u>derivative</u> of f at x_0 and is denoted by $f'(x_0)$:

$$\boxed{f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}\ .}$$

Note. If we pose $x = x_0 + h$, we can also write

$$\boxed{f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}\ .}$$

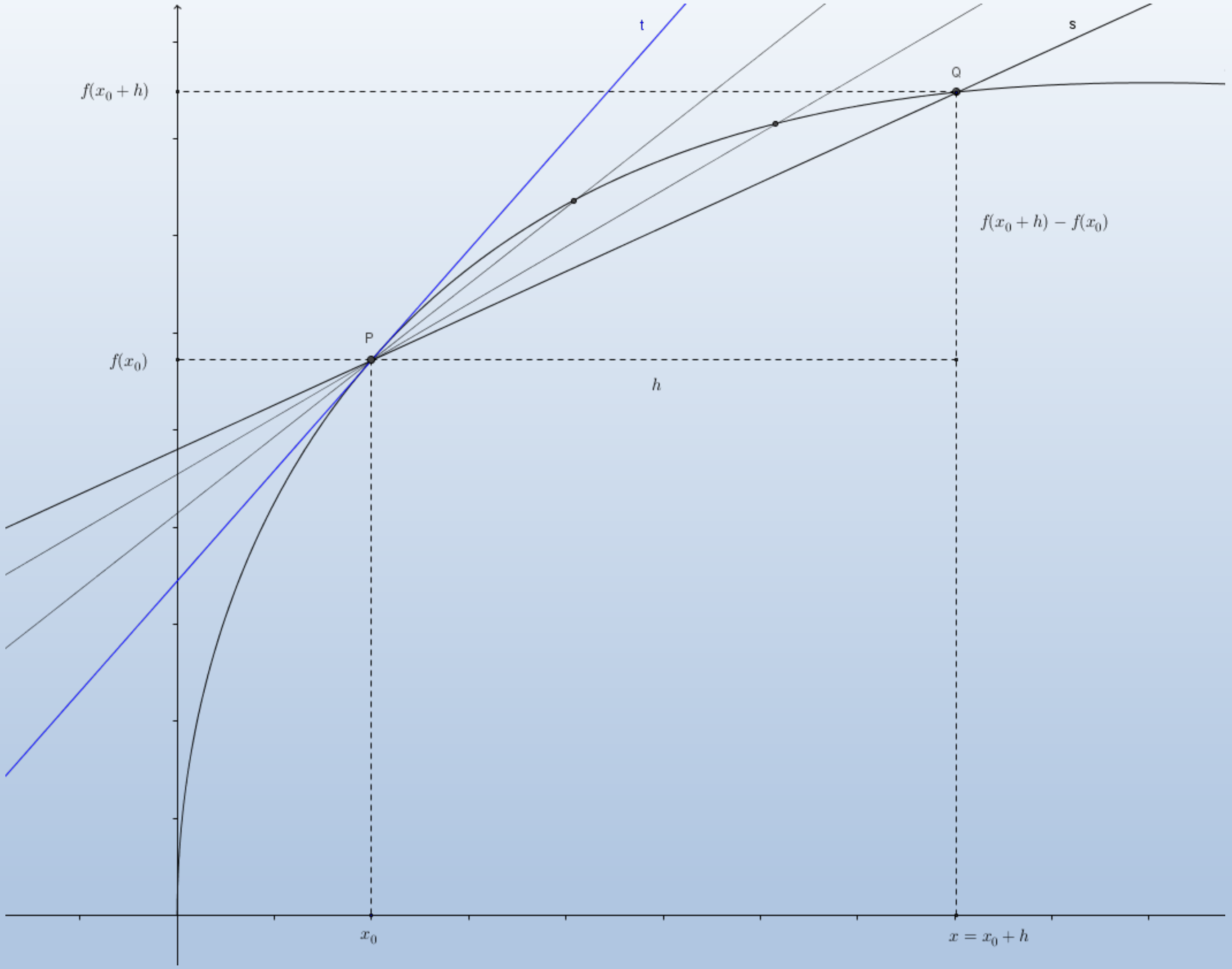
GEOMETRIC INTERPRETATION

The ratio $\frac{f(x_0+h)-f(x_0)}{h}$ is called the difference quotient of f at x_0.

The difference quotient of f at x_0 represents the gradient (or slope) m_s of the secant line to the graph of f through $P(x_0, f(x_0))$ and a point $Q(x_0+h, f(x_0+h))$.

As x approaches x_0, the secant line through P and Q tends to the tangent line at P and the difference quotient $\frac{f(x)-f(x_0)}{x-x_0}$ tends by definition to the derivative $f'(x_0)$, which therefore represents the gradient m_t of the tangent line to the graph of f at $P(x_0, f(x_0))$.

Interactive exercise: the difference quotient



$\frac{f(x_0+h)-f(x_0)}{h}$	$\longrightarrow$	$f'(x_0)$
$=$	as $h \to 0$	$=$
m_s	$\longrightarrow$	m_t

ONE-SIDED DERIVATIVES

- The limit

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h},$$

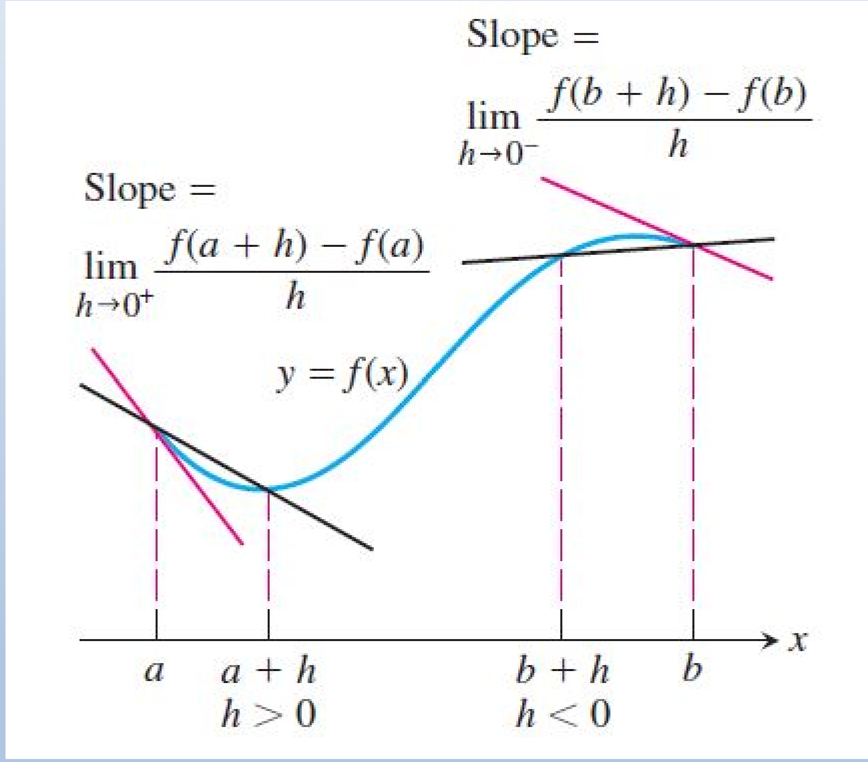
provided it exists, is called <u>the right-hand derivative</u> of f at x_0 and is denoted by $f'_+(x_0)$.

- In the same way, the limit

$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided it exists, is called <u>the left-hand derivative</u> of f at x_0 and is denoted by $f'_-(x_0)$.

- <u>A function is differentiable at x_0 if both right-hand and left-hand derivatives at x_0 exist, are finite and are equal.</u>



THE DERIVATIVE AS A FUNCTION

A function $f(x)$ is said to be <u>differentiable on a closed interval</u> $[a, b]$ if

- it is differentiable at every interior point of the interval, and
- there exist the right-hand derivative at the left endpoint a and the left-hand derivative at the right endpoint b and they are both finite.

In that case, we can consider the function $f': (a, b) \longrightarrow \mathbb{R}$, which assigns, to each $x \in (a, b)$, the derivative $f'(x)$ of f at x. This function is written $f'(x)$ and is called the <u>derivative function</u> or the derivative of f.

NOTATION

There are many ways to denote the derivative of a function $y = f(x)$. Beside $f'(x)$, the most common are:

f'	"f prime"	Nice and brief, but don't name the indipendent variable.
y'	"y prime"	
$\dfrac{dy}{dx}$	" $dy\ dx$" or "the derivative of y with respect to x"	Name both variables and uses d for derivative.
$\dfrac{df}{dx}$	" $df\ dx$" or "the derivative of f with respect to x"	
$\dfrac{d}{dx}f(x)$	" $d\ dx$ of f at x" or "the derivative of f at x"	Emphasize the idea that differentiation is an operation performed on f.
$Df(x)$	"the derivative of f at x"	

Second lesson

CALCULATING DERIVATIVES (I)

THE DERIVATIVE OF A CONSTANT FUNCTION AND OF THE IDENTITY FUNCTION

- **Constant function (constant rule)**

$$y = k \quad (k \in \mathbb{R}) \qquad \Longrightarrow \qquad y' = 0$$

Proof

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k-k}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

- **Identity function**

$$y = x \qquad \Longrightarrow \qquad y' = 1$$

Proof

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

THE DERIVATIVE OF A POWER FUNCTION (POWER RULE)

- **Positive integer power function**

$$y = x^n \quad (n \in \mathbb{N} \wedge n > 1) \quad \Longrightarrow \quad y' = nx^{n-1}$$

Proof

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{\left[x\left(1+\frac{h}{x}\right)\right]^n - x^n}{h} = \lim_{h\to 0} \frac{x^n\left(1+\frac{h}{x}\right)^n - x^n}{h} =$$

$$= \lim_{h\to 0} \frac{x^n\left[\left(1+\frac{h}{x}\right)^n - 1\right]}{h} = \lim_{h\to 0} \frac{x^{n-1}\left[\left(1+\frac{h}{x}\right)^n - 1\right]}{\frac{h}{x}} = x^{n-1} \lim_{h\to 0} \frac{\left[\left(1+\frac{h}{x}\right)^n - 1\right]}{\frac{h}{x}} = x^{n-1}\cdot n = nx^{n-1}$$

- **Real power function**

$$y = x^\alpha \quad (\alpha \in \mathbb{R}) \quad \Longrightarrow \quad y' = \alpha x^{\alpha-1}$$

THE DERIVATIVE OF A POWER FUNCTION (POWER RULE)

- **Special cases**

$y = x^2$	$\Longrightarrow$	$y' = 2x$
$y = \frac{1}{x} = x^{-1}$	$\Longrightarrow$	$y' = -x^{-2} = -\frac{1}{x^2}$
$y = \sqrt{x} = x^{\frac{1}{2}}$	$\Longrightarrow$	$y' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
$y = \sqrt[3]{x} = x^{\frac{1}{3}}$	$\Longrightarrow$	$y' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

LINEARITY OF DIFFERENTIATION

- **The constant multiple rule**

$y = k \cdot f(x) \quad (k \in \mathbb{R})$	$\Longrightarrow$	$y' = k \cdot f'(x)$

- **The sum rule**

$y = f(x) + g(x)$	$\Longrightarrow$	$y' = f'(x) + g'(x)$

- **Examples**

- $y = 2x^3 - 5x^2 + 3x - 2 \quad \Longrightarrow \quad y' = 6x^2 - 10x + 3$
- $y = 3\sqrt{x} - \frac{1}{\sqrt{x}} - \frac{2}{x} + \frac{1}{x^2} = 3x^{\frac{1}{2}} - x^{-\frac{1}{2}} - 2x^{-1} + x^{-2}$

$$y' = \frac{3}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}} + 2x^{-2} - 2x^{-3} = \frac{3}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}} + \frac{2}{x^2} - \frac{2}{x^3}$$

Third lesson

CALCULATING DERIVATIVES (II)

THE DERIVATIVE OF AN EXPONENTIAL FUNCTION

- **Exponential function**

$$y = a^x \quad (a > 0 \wedge a \neq 1) \qquad \Longrightarrow \qquad y' = a^x \ln a$$

Proof

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h} = \lim_{h\to 0} \frac{a^x\left(a^h - 1\right)}{h} = a^x \lim_{h\to 0} \frac{\left(a^h - 1\right)}{h} = a^x \ln a$$

- **Special case**

$$y = e^x \qquad \Longrightarrow \qquad y' = e^x$$

THE DERIVATIVE OF A LOGARITHMIC FUNCTION

- **Logarithmic function**

$$y = \log_a x \quad (a > 0 \wedge a \neq 1) \quad \Longrightarrow \quad y' = \frac{1}{x}\log_a e$$

Proof

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h} = \lim_{h\to 0}\frac{\log_a\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0}\frac{\log_a\left(1+\frac{h}{x}\right)}{h} = \lim_{h\to 0}\frac{1}{x}\frac{\log_a\left(1+\frac{h}{x}\right)}{\frac{h}{x}} = \frac{1}{x}\log_a e$$

- **Special case**

$$y = \ln x \quad \Longrightarrow \quad y' = \frac{1}{x}$$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

- **Sinus function**

$$y = \sin x \qquad \Longrightarrow \qquad y' = \cos x$$

Proof

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} =$$

$$= \lim_{h\to 0}\frac{\sin x\,(\cos h-1)+\cos x\sin h}{h} = \lim_{h\to 0}\left(\sin x\frac{\cos h-1}{h}+\cos x\frac{\sin h}{h}\right) = \sin x\cdot 0+\cos x\cdot 1 = \cos x$$

- **Cosinus function**

$$y = \cos x \qquad \Longrightarrow \qquad y' = -\mathrm{sin}x$$

DERIVATIVES OF PRODUCTS AND QUOTIENTS

- **The product rule**

$$y = f(x) \cdot g(x) \qquad \Longrightarrow \qquad y' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

- **Example** $\quad y = x^2 \ln x \quad \Longrightarrow \quad y' = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x = x(2 \ln x + 1)$

- **The quotient rule**

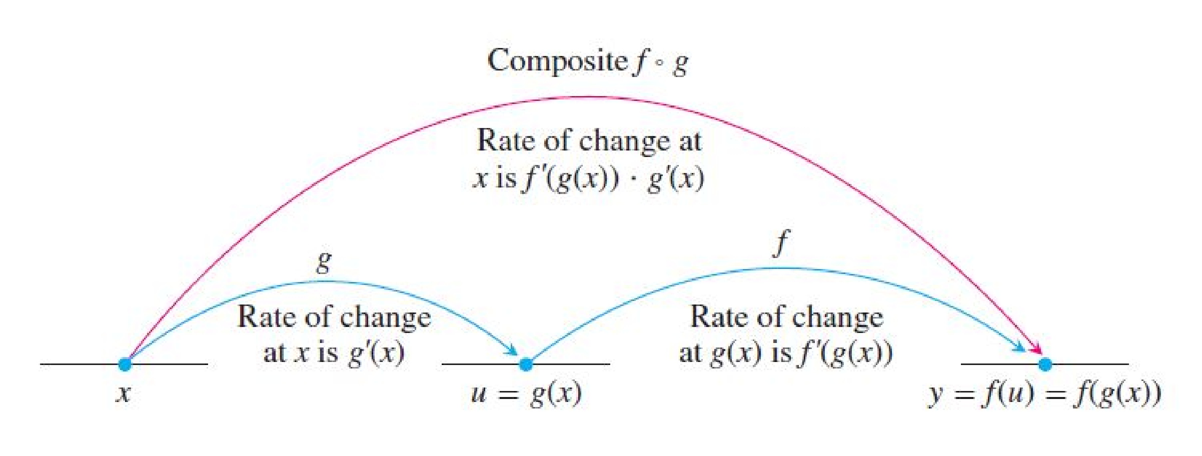
$$y = \frac{f(x)}{g(x)} \qquad \Longrightarrow \qquad y' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

- **Examples**
- $y = \tan x = \frac{\sin x}{\cos x} \quad \Longrightarrow \quad y' = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = 1 + \tan^2 x$
- $y = \cot x = \frac{\cos x}{\sin x} \quad \Longrightarrow \quad y' = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -1 - \cot^2 x$

THE DERIVATIVE OF A COMPOSITE FUNCTION

- **The chain rule or «outside-inside» rule**

$$y = (f \circ g)\,(x) = f(g(x)) \qquad \Longrightarrow \qquad y' = f'(g(x)) \cdot g'(x)$$



- **Examples**

- $y = \ln \sin x \quad \Longrightarrow \quad y' = \frac{1}{\sin x} \cdot \cos x = \cot x$
- $y = \cos^2 x = (\cos x)^2 \quad \Longrightarrow \quad y' = 2\cos x\,(-\sin x) = -2\sin x \cos x = -\sin 2x$

Fourth lesson

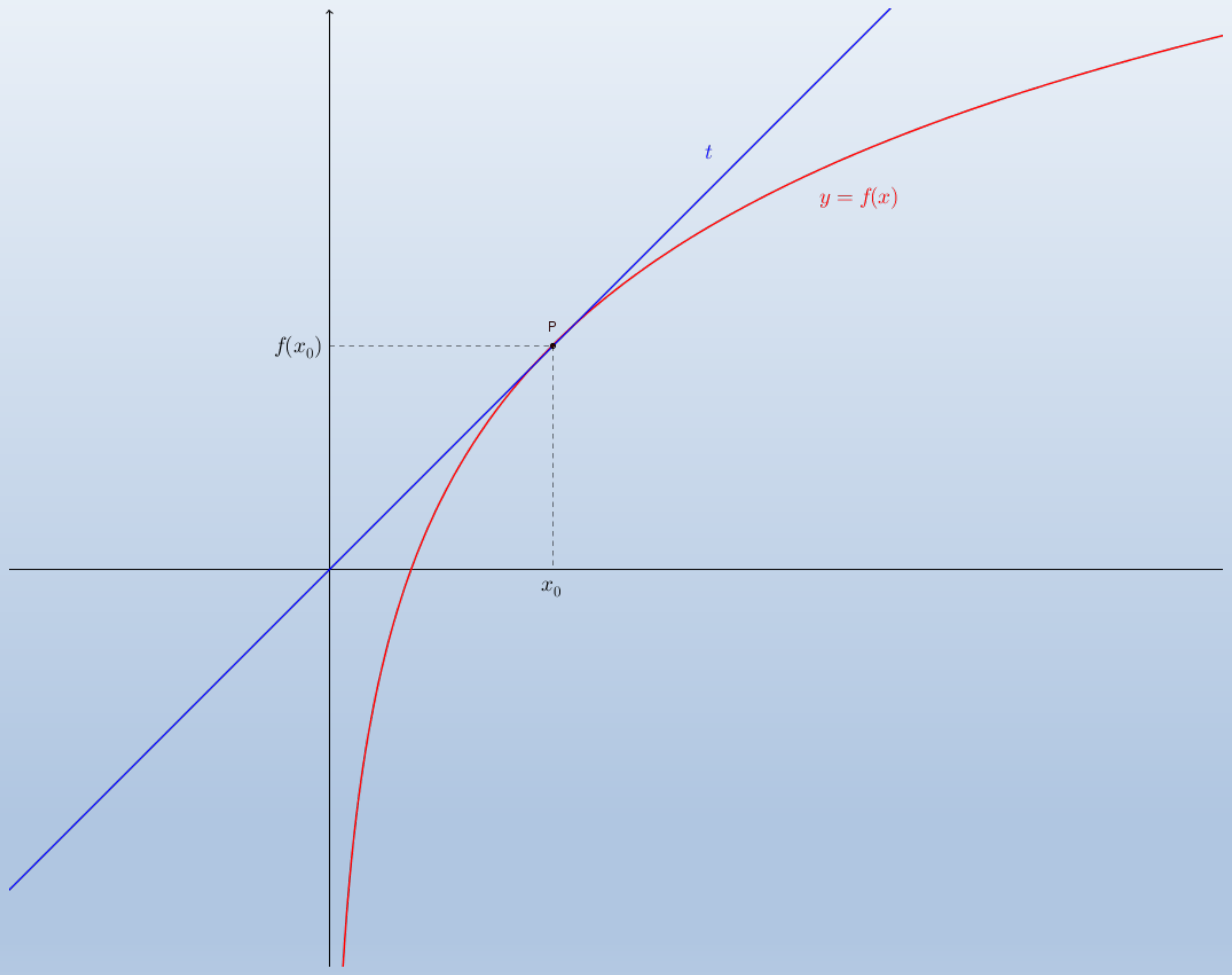
MORE ON DIFFERENTIABILITY

EQUATION OF THE TANGENT LINE

Suppose that f is differentiable at x_0.

Then the equation of the tangent line to the graph of f at $P(x_0, f(x_0))$ is

$$\boxed{y - f(x_0) = f'(x_0) \cdot (x - x_0).}$$



EQUATION OF THE TANGENT LINE

Example

$y = \ln x$

$x_0 = 1$

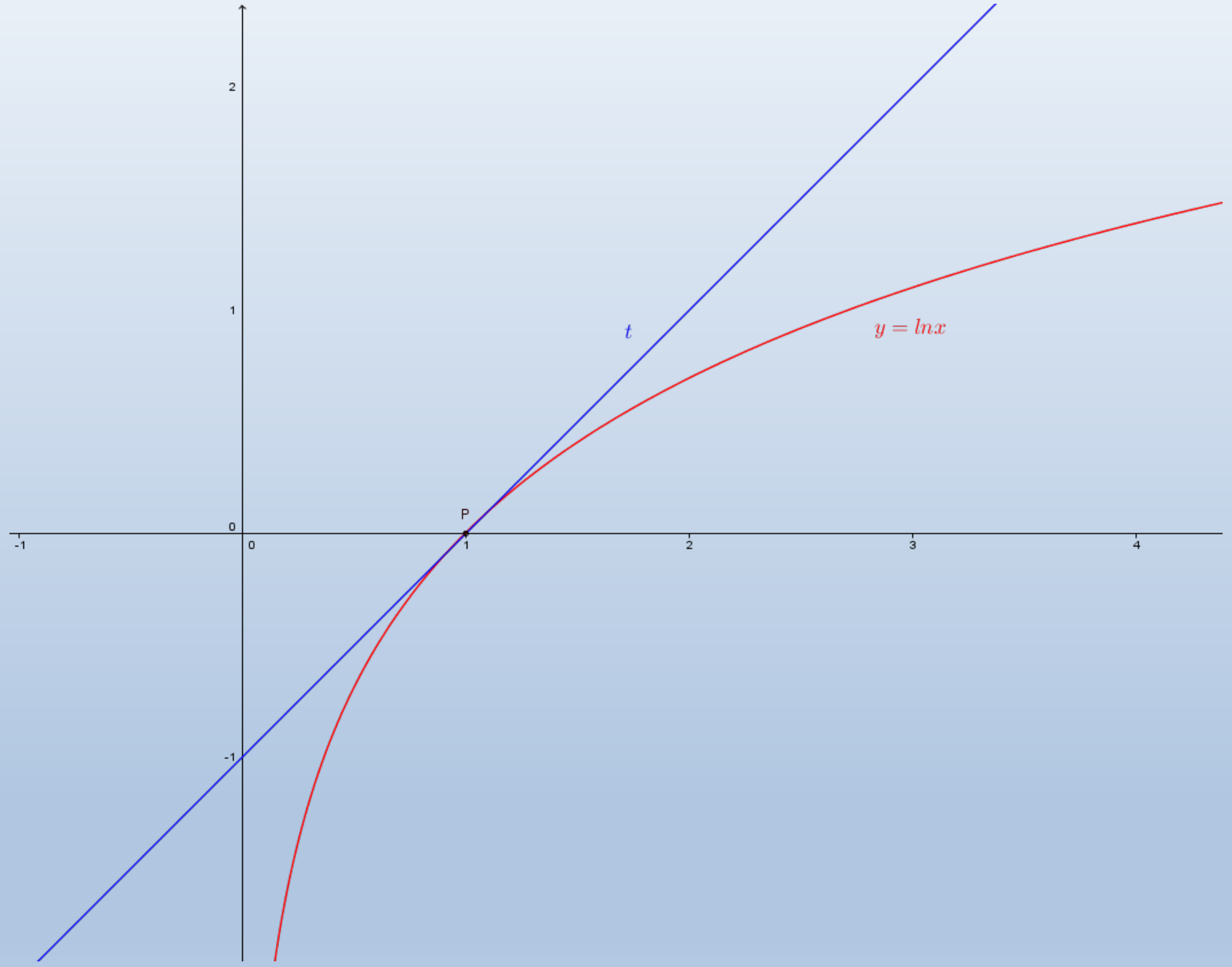
$f(x_0) = f(1) = 0$

$y' = \frac{1}{x}$

$f'(1) = 1$

$t\colon\ y - 0 = 1(x - 1)$

$t\colon\ y = x - 1$



NON-DIFFERENTIABLE FUNCTIONS

A function will not have a derivative at a point x_0 where the slopes of the secant lines

$$\frac{f(x) - f(x_0)}{x - x_0}$$

fail to approach a limit as x approaches x_0.

The next figures illustrate three different instances where this occurs.

For example, a function whose graph is otherwise smooth will fail to have a derivative at a point where the graph has …

… A CORNER,

where the one-sided derivatives exist, are finite but differ:

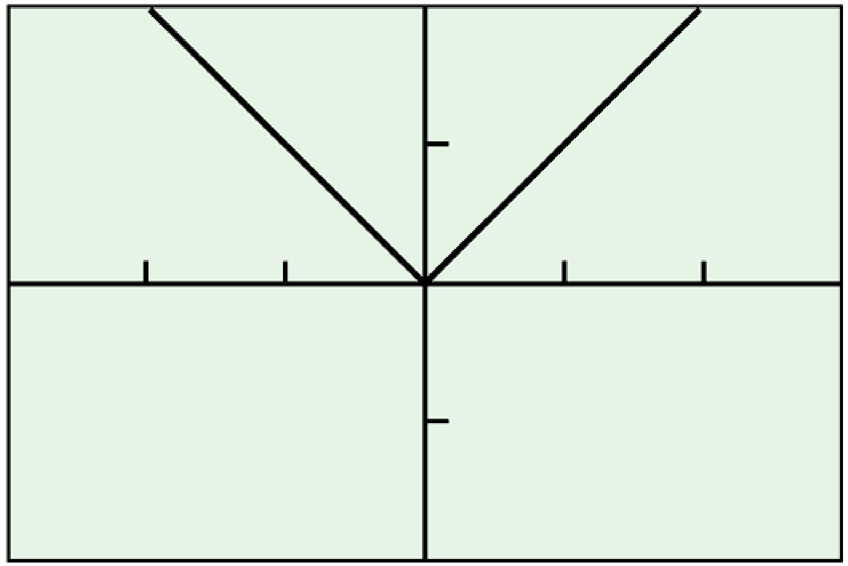
$$f'_-(x_0) \neq f'_+(x_0).$$

Example 1

$f(x) = |x|$ at $x_0 = 0.$

$f'_-(0) = -1$

$f'_+(x_0) = 1$



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… A CORNER

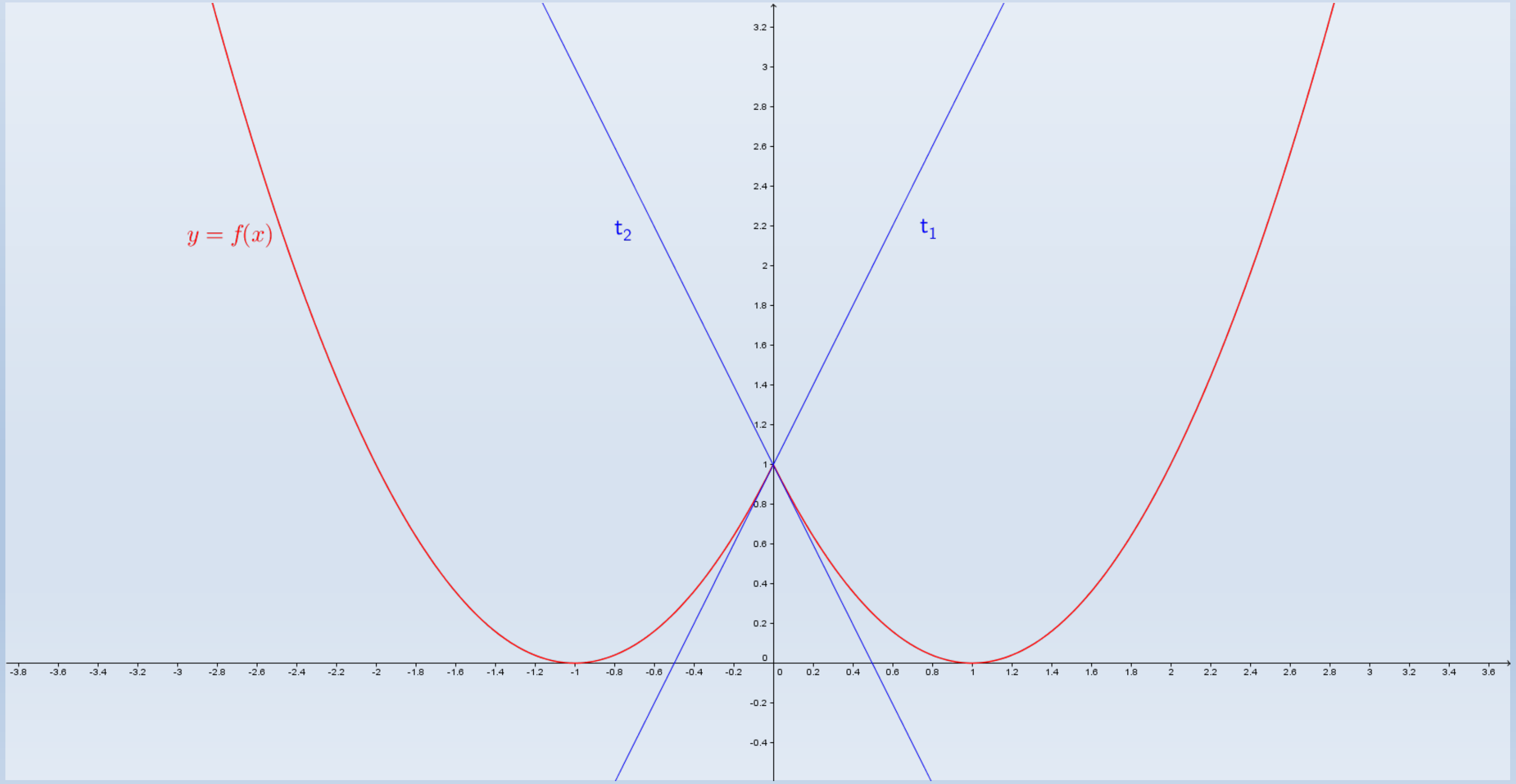
Example 2

$$f(x) = \begin{cases} (x+1)^2, \text{for } x < 0 \\ (x-1)^2, \text{for } x \geq 0 \end{cases}$$

at $x_0 = 0.$

$$f'_{-}(0) = -1$$

$$f'_{+}(x_0) = 1$$



… A CUSP,

where the one-sided derivatives exist and are infinite and of opposite sign:

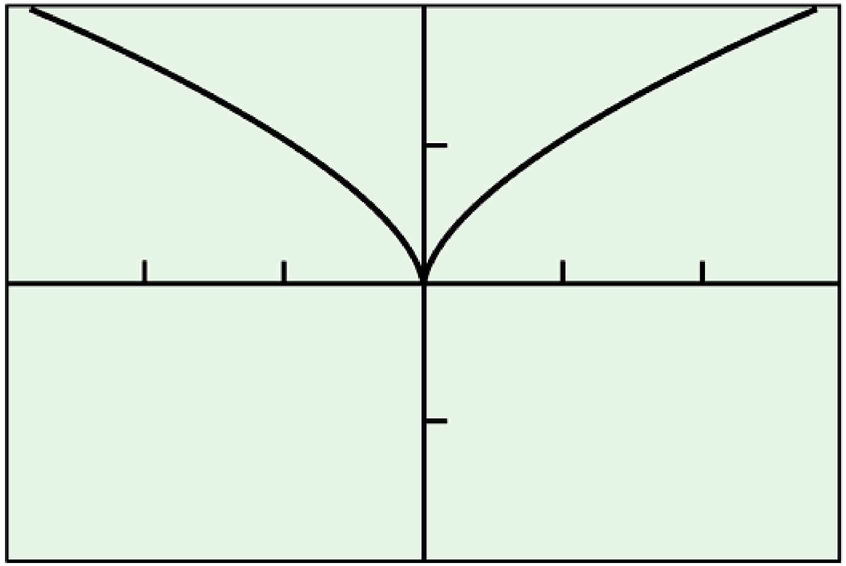
either $f'_-(x_0) = -\infty \ \wedge \ f'_+(x_0) = +\infty$

or $\qquad f'_-(x_0) = +\infty \ \wedge \ f'_+(x_0) = -\infty.$

Example

$f(x) = \sqrt[3]{x^2}$ at $x_0 = 0.$

$f'_-(x_0) = -\infty \ \wedge \ f'_+(x_0) = +\infty$



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… A VERTICAL TANGENT,

where the one-sided derivatives exist and are infinite and of the same signs:

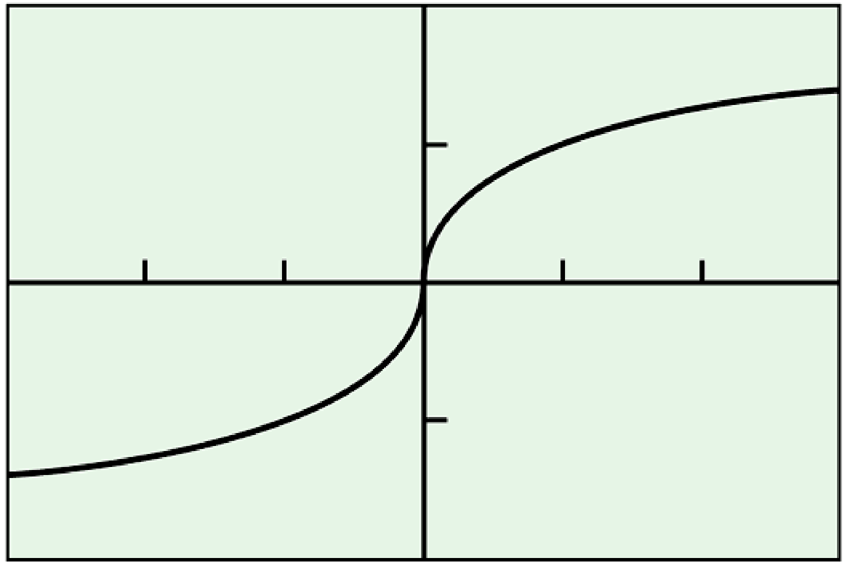
either $f'_-(x_0) = f'_+(x_0) = +\infty$

or $f'_-(x_0) = f'_+(x_0) = -\infty.$

Example

$f(x) = \sqrt[3]{x}$ at $x_0 = 0.$

$f'_-(x_0) = f'_+(x_0) = +\infty$



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DIFFERENTIABILITY IMPLIES CONTINUITY

Theorem

If f is differentiable at x_0, then f is continuous at x_0.

Proof

$$\lim_{x\to x_0}[f(x)-f(x_0)] = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\cdot(x-x_0) =$$

$$= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\cdot\lim_{x\to x_0}(x-x_0) = f'(x_0)\cdot 0 = 0.$$

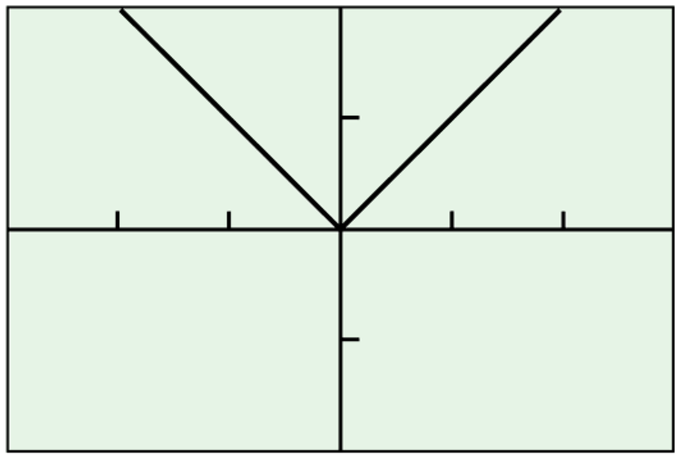
Therefore $\lim_{x\to x_0} f(x) = f(x_0)$ and f is continuous at x_0.

CONTINUITY DOESN'T IMPLY DIFFERENTIABILITY

The converse of the previous theorem is false.

A continuous function might have a corner, a cusp or a vertical tangent line, and hence not be differentiable at a given point.

All the functions considered in the previous examples are continuous, but not differentiable at $x_0 = 0$.



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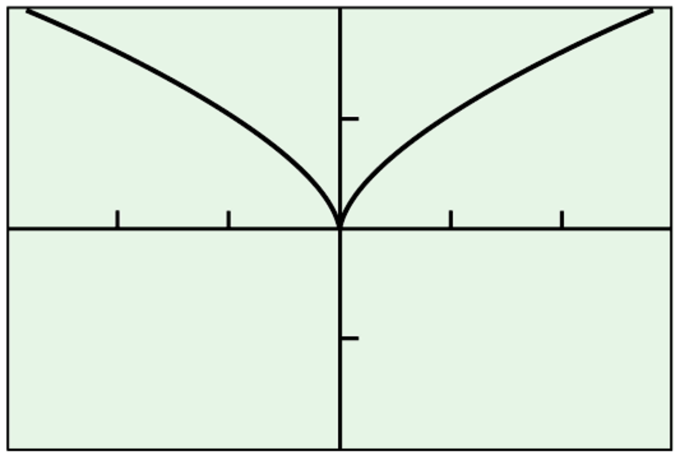

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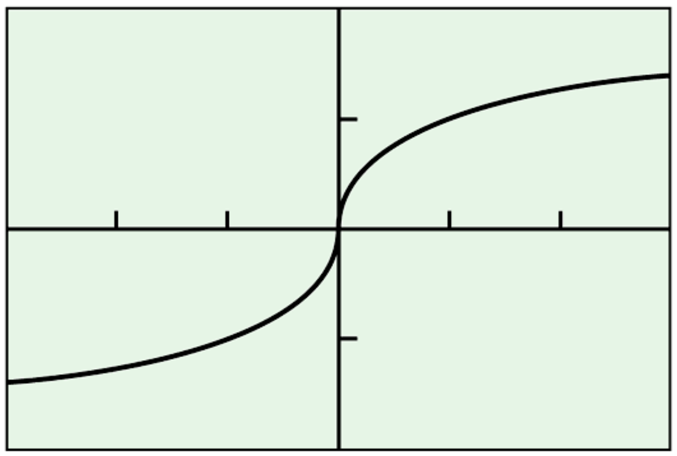

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